Combinatorial and Algorithmic Aspects of Hyperbolic Polynomials

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Abstract

Let $p(x_1, ..., x_n) = \sum_{(r_1,...,r_n) \in I_{n,n}} a_{(r_1,...,r_n)} \prod_{1 \leq i \leq n} x_i^{r_i}$ be homogeneous polynomial of degree n in n real variables with integer nonnegative coefficients. The support of such polynomial $p(x_1, ..., x_n)$ is defined as $supp(p) = \{(r_1, ..., r_n) \in I_{n,n} : a_{(r_1,...,r_n)} \neq 0\}$. The convex hull $CO(supp(p))$ of $supp(p)$ is called the Newton polytope of p . We study the following decision problems , which are far-reaching generalizations of the classical perfect matching problem :

- **Problem 1 .** Consider a homogeneous polynomial $p(x_1, ..., x_n)$ of degree n in n real variables with nonnegative integer coefficients given as a black box (oracle) . *Is it true that* $(1, 1, .., 1) \in supp(p)$ *?*
- **Problem 2 .** Consider a homogeneous polynomial $p(x_1, ..., x_n)$ of degree n in n real variables with nonnegative integer coefficients given as a black box (oracle) . *Is it true that* $(1, 1, .., 1) \in CO(supp(p))$ *?*

We prove that for hyperbolic polynomials these two problems are equivalent and can be solved by deterministic polynomial-time oracle algorithms . This result is based on a "hyperbolic" generalization of the Rado theorem . We also present combinatorial and algebraic applications of this "hyperbolic" generalization of the Rado theorem (prove that the support $supp(p)$ of P-hyperbolic polynomial p is an intersection of some Integral Polymatroid with the hyperplane $\{(r_1, ..., r_n) : \sum_{1 \leq i \leq n} r_i = n\}$) and pose some open problems.

1 Introduction and motivating examples

The layout of the paper :

We introduce the main topics and motivations in Section 1 . In Section 1.1 we present a naive algorithm to solve Problem 1 in the general case . We show in Appendix **D** *that this algorithm is , in a sense , optimal .*

(Incidentally (or not) , the situation here is very similar with the optimality of the square root in the famous quantum Grover's search algorithm .)

In Section 1.2 we remind the basic properties of hyperbolic polynomials used in this paper .

In Section 2 we state a hyperbolic analogue of the Rado theorem (Theorem 2.2) , which is the main mathematical result of the paper . Theorem 2.2 sheds more light on the algebraic-geometric nature of such fundamental combinatorial results as Hall's and Rado's theorems .

In Section 2.1 we define and study doubly-stochastic polynomials (an useful generalization of standard doubly-stochastic matrices). We also state there a hyperbolic analogue of the van der Waerden conjecture .

In Section 3 we introduce and analyse the ellipsoid algorithm which solves Problem 1 and Problem 2 on the class of S-hyperbolic polynomials . The essence of the results in Section 3 is that once Hall's like conditions (the exponential number of them) are proved to be necessary and sufficient , they can be checked by a polynomial time deterministic oracle algorithms . The algorithm , which we use , is based not on the linear programming but on some rather nonlinear convex programs similar to considered in [23] , [21] , [22].

In section 4 we introduce and analyse another algorithm , which is a "polynomial" generalization of the Sinkhorn Scaling .

In Section 5 we use Theorem 2.2 to get a more refine (polymatroidal) properties of the supports of P-hyperbolic polynomials and explain how our results generalize the main result from [7].

In Section 6 we pose some open problems and share our enthusiasm about the topic .

The proofs of the main results are presented in Appendices **A,B,C,D** .

Let $p(x_1,...,x_n) = \sum_{(r_1,...,r_n) \in I_{n,n}} a_{(r_1,...,r_n)} \prod_{1 \leq i \leq n} x_i^{r_i}$ be homogeneous polynomial of degree n in n real variables. Here $I_{k,n}$ stands for the set of vectors $r = (r_1,...,r_k)$ with nonnegative integer components and $\sum_{1 \leq i \leq k} r_i = n$. In this paper we primarily study homogeneous polynomials with nonnegative integer coefficients .

Definition 1.1: The support of the polynomial $p(x_1,...,x_n)$ as above is defined as $supp(p) = \{(r_1,...,r_n) \in I_{n,n} : a_{(r_1,...,r_n)} \neq 0\}$. The convex hull $CO(supp(p))$ of $supp(p)$ is called the Newton polytope of p . ▮

We will study the following decision problems :

- **Problem 1 .** Consider a homogeneous polynomial $p(x_1,...,x_n)$ of degree n in n real variables with nonnegative integer coefficients given as a black box (oracle) . *Is it true that* $(1,1,..,1) \in supp(p)$ *?*
  An equivalent question is : *Is it true that* $\frac{\partial^n}{\partial x_1 ... \partial x_n} p(x_1,...,x_n) \neq 0$ *?*

- **Problem 2 .** Consider a homogeneous polynomial $p(x_1,...,x_n)$ of degree n in n real variables with nonnegative integer coefficients given as a black box (oracle) . *Is it true that* $(1,1,..,1) \in CO(supp(p))$ *?*

Our goal is solve these decision problems using deterministic polynomial-time oracle algorithms , i.e. algorithms which evaluate the given polynomial $p(.)$ at a number of rational vectors $(q_1,...,q_n)$ which is polynomial in n and $\log(p(1,1,..,1))$; these rational vectors $(q_1,...,q_n)$ are supposed to have bit-wise complexity which is polynomial in n and $\log(p(1,1,..,1))$; and the additional auxilary arithmetic computations also take a polynomial number of steps in n and $\log(p(1,1,..,1))$.

The next example explains some (well known) origins of the both problems .

Example 1.2: Consider first the following homogeneous polynomial from [28] : $p(x_1,...,x_n) = tr((D(x)A)^n)$, where $D(x)$ is a $n \times n$ diagonal matrix $Diag(x_1,...,x_n)$; and A is $n \times n$ matrix with $(0,1)$ entries , i.e. A is an adjacency matrix of some directed graph Γ . Clearly , this polynomial $p(x_1,...,x_n)$ has nonnegative integer coefficients . It was proved in [28] that $\frac{1}{n}\frac{\partial^n}{\partial x_1...\partial x_n} tr((D(x)A)^n)$ is equal to the number of Hamiltonian circuits in the graph Γ . Notice that the polynomial $tr(D(x)A)^n$ can be evaluated in $O(n^3 \log(n))$ arithmetic operations and $(1,1,...,1) \in supp(p)$ iff there exists a Hamiltonian circuit in the graph Γ. Also $\log(p(1,1,..,1)) \leq n\log(n)$. Therefore , unless $P = NP$, there is no hope to design deterministic polynomial-time oracle algorithm solving Problem 1 in this case . (The author is indebted to A.Barvinok for pointing out this polynomial .)
Consider , with the same adjacency matrix A , another homogeneous polynomial
$Mul(x_1,...,x_n) = \prod_{1\leq i\leq n}\sum_{1\leq i\leq n} A(i,j)x_j$. Then
$\frac{\partial^n}{\partial x_1...\partial x_n} Mul(x_1,...,x_n) = Per(A)$.
Therefore for the multilinear polynomial $Mul(x_1,...,x_n)$ Problem 1 is a "black box" analogue of checking the existence of the perfect bipartite matching .

Next consider the following class of determinantal polynomials :

$$q(x_1,...,x_n) = \det(\sum_{1\leq i\leq n} A_i x_i),$$

where $\mathbf{A} = (A_1,...,A_n)$ is a n-tuple of positive semidefinite $n \times n$ hermitian matrices , i.e. $A_i \succeq 0$, with integer entries . Recall that the mixed discriminant

$$D(\mathbf{A}) = \frac{\partial^n}{\partial \alpha_1...\partial \alpha_n}\det(\sum_{1\leq i\leq n} A_i x_i).$$

(If the matrices A_i above are diagonal , i.e. $A_i = Diag(b_{i,1},...,b_{i,n}), 1 \leq i \leq n$, then the mixed discriminant is reduced to the permanent : $D(\mathbf{A}) = Per(B), B = \{b_{i,j}; 1 \leq i,j \leq n$).

It is well known (see , for instance , [22]) that a determinantal polynomial $q(.)$ can be represented as

$$q(x_1,..,x_n) = \sum_{r\in I_{n,n}} \prod_{1\leq i\leq n} x_i^{r_i} D(\mathbf{A}_r)\frac{1}{\prod_{1\leq i\leq n} r_i!}, \tag{1}$$

where a n-tuple $\mathbf{A}_r$ of square matrices consists of r_i copies of $A_i, 1 \leq i \leq k$. One of the equivalent formulations [34] of the classical Rado theorem states that $D(\mathbf{A}_{(1,1,...,1)}) > 0$ iff

$$Rank(\sum_{i\in S} A_i) \geq |S| \text{ for all } S \subset \{1,2,...,n\} \tag{2}$$

(The diagonal case is the famous Hall's theorem on the perfect bipartite matchings .)
The Rado theorem is just a particular case of famous Edmonds theorem on the rank of intersection of two matroids . Therefore , given a n-tuple $\mathbf{A} = (A_1,...,A_n)$ of positive semidefinite $n \times n$ hermitian matrices , one can decide in deterministic polynomial time if $D(\mathbf{A}) > 0$. We

will explain below that this decision problem can be solved by a deterministic polynomial-time oracle algorithm . I.e. we only use some values of $\det(\sum_{1 \leq i \leq n} A_i x_i)$ without reconstructing the actual tuple $\mathbf{A} = (A_1, ..., A_n)$.

The natural question , in our opinion , is which algebraic-geometric properties make the class of determinantal polynomials "easy" and the class of Barvinok's polynomials $tr(D(x)A)^n$ "hard" . This paper suggests one answer to the question.

One important corollary of the Rado conditions (2) is that

$$supp(q) = CO(supp(q)) \cap I_{n,n}. \tag{3}$$

I.e. if integer vectors $r, r(1), r(2), ..., r(k) \in I(n,n)$ and

$$r = \sum_{1 \leq i \leq k} a(i) r(i), a(i) \geq 0, 1 \leq i \leq k; \sum_{1 \leq i \leq k} a(i),$$

and $D(\mathbf{A}_{r(i)}) > 0, 1 \leq i \leq k$ then also $D(\mathbf{A}_r) > 0$. Notice that in this case Problem 1 and Problem 2 are equivalent .
We can rewrite Rado conditions (2) as follows :

$$\max_{r \in supp(q)} \sum_{i \in S} r_i \geq |S| \text{ for all } S \subset \{1, 2, ..., n\} \tag{4}$$

Putting things together we get the following Fact .

Fact 1.3: The following properties of determinantal polynomial $q(x_1, ..., x_n) = \det(\sum_{1 \leq i \leq n} A_i x_i)$ with $n \times n$ hermitian matrices $A_i \succeq 0, 1 \leq i \leq n$ are equivalent .

1. $(1, 1, .., 1) \notin supp(q)$.
2. $(1, 1, .., 1) \notin CO(supp(q))$.
3. There exists nonempty $S \subset \{1, 2, ..., n\}$ such that

$$\sum_{1 \leq i \leq n} r_i s_i < \sum_{1 \leq i \leq n} s_i = |S| \text{ for all} (r_1, ..., r_n) \in supp(q), \tag{5}$$

   where $(s_1, ..., s_n)$ is a characteristic function of the subset S , i.e. $s_i = 1$ if $i \in S$, and $s_i = 0$ otherwise .

   Notice that if (5) holds then the distance $dist(e, CO(supp(q)))$ from the vector $e = (1, ..., 1)$ to the Newton polytope $CO(supp(q))$ is at least $\sqrt{\frac{n}{|S|(n-|S|)}} \geq \frac{2}{\sqrt{n}}$.

∎

We will show that for any class of polynomials satisfying Fact 1.3 there exists a deterministic polynomial-time oracle algorithm solving both Problem 1 and Problem 2 , which are , of course , equivalent in this case . Our algorithm is based on the reduction to some convex programming problem and the consequent use of the Ellipsoids method .
The next fact about determinantal polynomials , namely their hyperbolicity , is "responsible" for Fact 1.3 .

Fact 1.4: Consider a determinantal polynomial $q((x_1,...,x_n) = \det(\sum_{1 \leq i \leq n} A_i x_i)$ with $A_i \succeq 0, 1 \leq i \leq n$. Assume that q is not identically zero , i.e. that $B =: \sum_{1 \leq i \leq n} A_i \succ 0$ (the sum is strictly positive definite). For a real vector $(x_1,...,x_n) \in R^n$ consider the following polynomial equation of degree n in one variable :

$$P(t) = q(x_1 - t, x_2 - t, ..., x_n - t) = \det(\sum_{1 \leq i \leq n} A_i x_i - t \sum_{1 \leq i \leq n} A_i) = 0. \quad (6)$$

Equation (6) has n real roots roots counting the multiplicities ; if the real vector $(x_1,...,x_n) \in R^n$ has nonnegative entries then all roots of (6) are nonnegative real numbers . ∎

The main result of this paper that this hyperbolicity , which we will describe formally in Section 1.1 , is sufficient for Fact 1.3 ; i.e. Fact 1.4 implies Fact 1.3 . ∎

1.1 "Naive" algorithms

One possible "naive" algorithm to solve Problem 1 is just to compute $\frac{\partial^n}{\partial x_1 ... \partial x_n} p(x_1,...,x_n)$. Recall that the number of coefficients of a homogeneous polynomial of degree n in n real variables is equal to $\frac{(2n-1)!}{n!(n-1)!} \approx 2^{2n}$. We can compute all the coefficients of $p(x_1,...,x_n)$ via standard multidimensional interpolation , but this interpolation will need $\frac{(2n-1)!}{n!(n-1)!} \approx 2^{2n}$ oracle calls . There is an algorithm which computes $\frac{\partial^n}{\partial x_1 ... \partial x_n} p(x_1,...,x_n)$ using only 2^{n-1} oracle calls :

$$\frac{\partial^n}{\partial x_1 ... \partial x_N} p(x_1,...,x_n) = 2^{-n+1} \sum_{b_i \in \{-1,+1\}, 2 \leq i \leq n} p(1, b_2, ..., b_n) \prod_{2 \leq i \leq n} b_i. \quad (7)$$

This formula is , in a sense , optimal . I.e. there exists a nearly matching lower bound . The corresponding result and connections to computations of the permanent are presented in Appendix D .

We will explain below that if p is a homogeneous polynomial of degree n in n real variables with nonnegative integer coefficients then $(1,1,..,1) \in CO(supp(p))$ iff $p(x_1,...,x_n) \geq \prod_{1 \leq i \leq n} x_i$ for all vectors $(x_1,...,x_n)$ with positive real coordinates . Therefore Problem 2 is equivalent to checking if the polynomial $P(y_1,...,y_n) = p(1+y_1^2,...,1+y_n^2) - \prod_{1 \leq i \leq n} 1 + y_i^2$ is nonnegative on R^n.

1.2 Hyperbolic polynomials

The following concept of hyperbolic polynomials was originated in the theory of partial differential equations [18], [9] ,[10] .
A homogeneous polynomial $p(x), x \in R^m$ of degree n in m real varibles is called hyperbolic in the direction $e \in R^m$ (or e- hyperbolic) if for any $x \in R^m$ the polynomial $p(x - \lambda e)$ in the one variable λ has exactly n real roots counting their multiplicities. We assume in this paper that $p(e) > 0$. Denote an ordered vector of roots of $p(x-\lambda e)$ as $\lambda(x) = (\lambda_1(x) \geq \lambda_2(x) \geq ...\lambda_n(x))$. It is well known that the product of roots is equal to $p(x)$. Call $x \in R^m$ e-positive (e-nonnegative) if $\lambda_n(x) > 0$ ($\lambda_n(x) \geq 0$). The fundamental result [18] in the theory of hyperbolic polynomials states that the set of e-nonnegative vectors is a closed convex cone. A k-tuple of vectors $(x_1, ...x_k)$ is called e-positive (e-nonnegative) if $x_i, 1 \leq i \leq k$ are e-positive (e-nonnegative). We denote the closed convex cone of e-nonnegative vectors as $N_e(p)$, and the open convex cone of e-positive vectors as $C_e(p)$.

Recent interest in the hyperbolic polynomials got sparked by the discovery [12] ,[11] that $\log(p(x))$ *is a self-concordant barrier for the opened convex cone* $C_e(p)$ *and therefore the powerful mashinery of interior-point methods can be applied . It is an important open problem whether this cone* $C_e(p)$ *has a semi-definite representation .*

It has been shown in [18] (see also [26]) that an e- hyperbolic polynomial p is also d-hyperbolic for all e-positive vectors $d \in C_e(p)$.
Let us fix n real vectors $x_i \in R^m, 1 \leq i \leq n$ and define the following homogeneous polynomial:

$$P_{x_1,..,x_n}(\alpha_1, ..., \alpha_n) = p(\sum_{1 \leq i \leq n} \alpha_i x_i) \tag{8}$$

Following [26] , we define the p-mixed form of an n-vector tuple $\mathbf{X} = (x_1, .., x_n)$ as

$$M_p(\mathbf{X}) =: M_p(x_1, .., x_n) = \frac{\partial^n}{\partial \alpha_1 ... \partial \alpha_n} p(\sum_{1 \leq i \leq n} \alpha_i x_i) \tag{9}$$

Equivalently, the p-mixed form $M_p(x_1, .., x_n)$ can be defined by the polarization (see [26]) :

$$M_p(x_1, .., x_n) = 2^{-n} \sum_{b_i \in \{-1,+1\}, 1 \leq i \leq n} p(\sum_{1 \leq i \leq n} b_i x_i) \prod_{1 \leq i \leq n} b_i \tag{10}$$

Associate with any vector $r = (r_1, ..., r_n) \in I_{n,n}$ an n-tuple of m-dimensional vectors $\mathbf{X}_r$ consisting of r_i copies of $x_i (1 \leq i \leq n)$. It follows from the Taylor's formula that

$$P_{x_1,..,x_n}(\alpha_1, ..., \alpha_n) = \sum_{r \in I_{n,n}} \prod_{1 \leq i \leq n} \alpha_i^{r_i} M_p(\mathbf{X}_r) \frac{1}{\prod_{1 \leq i \leq n} r_i!} \tag{11}$$

For an e-nonnegative tuple $\mathbf{X} = (x_1, .., x_n)$, define its capacity as:

$$Cap(\mathbf{X}) = \inf_{\alpha_i > 0, \prod_{1 \leq i \leq n} \alpha_i = 1} P_{x_1,..,x_n}(\alpha_1, ..., \alpha_n) \tag{12}$$

Probably the best known example of a hyperbolic polynomial comes from the hyperbolic geometry :

$$P(\alpha_0, ..., \alpha_k) = \alpha_0^2 - \sum_{1 \leq i \leq k} \alpha_i^2 \tag{13}$$

This polynomial is hyperbolic in the direction $(1, 0, 0, ..., 0)$. Another "popular" hyperbolic polynomial is $\det(X)$ restricted on a linear real space of hermitian $n \times n$ matrices . In this case mixed forms are just mixed discriminants , hyperbolic direction is the identity matrix I , the corresponding closed convex cone of I-nonnegative vectors coincides with a closed convex cone of positive semidefinite matrices .
Less known , but very interesting , hyperbolic polynomial is the Moore determinant $Det_{(M)}(Y)$ restricted on a linear real space of hermitian quaternionic $n \times n$ matrices . The Moore determinant is , essentially , the Pfaffian (see the corresponding definitions and the theory in a very readable paper [38]) .

We use in this paper the following class of hyperbolic in the direction $(1, 1, ..., 1)$ polynomials of degree k :
$Q(\alpha_1, ..., \alpha_k) = M_p(\sum_{1 \leq i \leq k} \alpha_i x_i, ..., \sum_{1 \leq i \leq k} \alpha_i x_i, x_{k+1}, ..., x_n)$, where p is a e-hyperbolic polynomial of degree $n > k$, $(x_1, .., x_n)$ is e-nonnegative tuple , and the p-mixed form $M_p(\sum_{1 \leq i \leq k} x_i, ..., \sum_{1 \leq i \leq k} x_i, x_{k+1}, ..., x_n) > 0$.

We make a substantial use of the following very recent result [27] , which is a rather direct corollary of [1] , [37] .

Theorem 1.5: *Consider a homogeneous polynomial $p(y_1, y_2, y_3))$ of degree n in 3 real variables which is hyperbolic in the direction $(0, 0, 1)$. Assume that $p(0, 0, 1) = 1$. Then there exists two $n \times n$ real symmetric matrices A, B such that*

$$p(y_1, y_2, y_3)) = \det(y_1 A + y_2 B + y_3 I).$$

It has been shown in [19] that most of known facts , and some opened problems as well , about hyperbolic polynomials follow from Theorem 1.5 .

2 A hyperbolic analogue of the Rado theorem

Definition 2.1: Consider a homogeneous polynomial $p(x), x \in R^m$ of degree n in m real variables which is hyperbolic in the direction e.Denote an ordered vector of roots of $p(x - \lambda e)$ as $\lambda(x) = (\lambda_1(x) \geq \lambda_2(x) \geq ...\lambda_n(x))$. We define the p-rank of $x \in R^m$ in direction e as $Rank_p(x) = |\{i : \lambda_i(x) \neq 0\}|$. It follows from Theorem 1.5 that the p-rank of $x \in R^m$ in any direction $d \in C_e$ is equal to the p-rank of $x \in R^m$ in direction e , which we call the p-rank of $x \in R^m$. ∎

Consider the following polynomial in one variable $D(t) = p(td + x) = \sum_{0 \leq i \leq n} c_i t^i$. It follows from the identity (11) that

$$c_n = M_p(d,..,d)(n!)^{-1} = p(d), \qquad (14)$$
$$c_{n-1} = M_p(x,d,..,d)(1!(n-1)!)^{-1}, ...,$$
$$c_0 = M_p(x,..,x)(n!)^{-1} = p(x).$$

Let $(\lambda_1^{(d)}(x) \geq \lambda_2^{(d)}(x) \geq ... \geq \lambda_n^{(d)}(x))$ be the (real) roots of x in the e-positive direction d, i.e. the roots of the equation $p(td - x) = 0$. Define (canonical symmetric functions) :

$$S_{k,d}(x) = \sum_{1 \leq i_1 < i_2 < ... < i_k \leq n} \lambda_{i_1}(x)\lambda_{i_2}(x)...\lambda_{i_k}(x).$$

Then $S_{k,d}(x) = \frac{c_{n-k}}{c_n}$. Clearly if x is e-nonnegative then for any e-positive vector d the p-rank $Rank_p(x) = \max\{k : S_{k,d}(x) > 0\}$. The next theorem , which we prove in Appendix **A** , is the main mathematical result of this paper . Our main tool is Theorem 1.5 , which facilites a rather easy induction . We also use a particularly easy case of the Rado theorem (see Remark A.5 for the details) .

Theorem 2.2: *Consider a homogeneous polynomial $p(x), x \in R^m$ of degree n in m real variables which is hyperbolic in the direction e , $p(e) > 0$. Let $\mathbf{X} = (x_1, ...x_n), x_i \in R^m$ be e-nonnegative n-tuple of m-dimensional vectors , i.e. $x_i, 1 \leq i \leq n$ are e-nonnegative .*
Then the p-mixed form $M_p(\mathbf{X}) =: M_p(x_1,..,x_n)$ is positive iff the following generalized Rado conditions hold :

$$Rank_p(\sum_{i \in S} x_i) \geq |S| \quad for\ all \quad S \subset \{1,2,...,n\}. \qquad (15)$$

Definition 2.3: Call a homogeneous polynomial $p(\alpha), \alpha \in R^n$ of degree n in n real variables P-hyperbolic if it is hyperbolic in direction $e = (1,1,...1)$ (vector of all ones) , $p(e) > 0$ and all the canonical orts $e_i, 1 \leq i \leq n$ (rows of the identity matrix I) are e-nonnegative . In other words ,a homogeneous polynomial $p(\alpha), \alpha \in R^n$ of degree n in n real variables is P-hyperbolic if it is e-hyperbolic and its closed cone of e-nonnegative vectors contains the nonnegative orthant $R^n_+ = \{(x_1,...,x_n) : x_i \geq 0, 1 \leq i \leq n\}$. It follows from [26] that the coefficients of$P$-hyperbolic polynomials are nonnegative real numbers .
(Notice that the class of P-hyperbolic polynomials coincides with the class of polynomials $P_{x_1,..,x_n}(\alpha_1,...,\alpha_n) = p(\sum_{1 \leq i \leq n} \alpha_i x_i)$, where p is e -hyperbolic polynomial of degree n in m real variables , a n-tuple $(x_1,..,x_n)$ of m-dimensional real vectors is e-nonnegative and $\sum_{1 \leq i \leq n} x_i$ is e-positive .)
Call a homogeneous polynomial $q(\alpha), \alpha \in R^n$ of degree n in n real variables with nonnegative coefficients S-hyperbolic if there exists a P-hyperbolic polynomial p such that $supp(p) = supp(q)$. ∎

(One natural class of S-hyperbolic polynomials ,not all of them P-hyperbolic , is $Vol(\alpha_1 X_1 + ...\alpha_n X_n)$, where X_i are convex compact subsets of R^n . See the corresponding not P-hyperbolic example in [26] .)

Corollary 2.4: *Let $q(\alpha), \alpha \in R^n$ be S-hyperbolic polynomial of degree n . Then $CO(supp(q)) \cap I_{n,n} = supp(q)$.*

Proof: It is enough to prove the corollary for P-hyperbolic polynomials. I.e. suppose that $q(\alpha_1, ..., \alpha_n) = p(\sum_{1 \leq i \leq n} \alpha_i x_i)$, where p is e -hyperbolic polynomial of degree n in m real variables , a n-tuple $(x_1, .., x_n)$ of m-dimensional real vectors is e-nonnegative and $\sum_{1 \leq i \leq n} x_i$ is e-positive . Then $r = (r_1, r_2, ..., r_n) \in supp(q)$ iff the p-mixed form $M_p(\mathbf{X}_r) > 0$, where the n-tuple $\mathbf{X}_r$ consists of r_i copies of $x_i, 1 \leq i \leq n$. Let $r^{(0)} = (r_1^{(0)}, ..., r_n^{(0)}) \in CO(supp(q))$. I.e. there exist $r^{(j)} \in supp(q), 1 \leq j \leq n$ such that $r^{(0)} = \sum_{1 \leq j \leq n} a_j r^{(j)}$ and $a_j \geq 0, \sum_{1 \leq j \leq n} a_j = 1$.

Let $r^{(j)} = (r_1^{(j)}, ..., r_n^{(j)}), 0 \leq j \leq n$. As $r^{(j)} \in supp(q), 1 \leq j \leq n$ thus $M_p(\mathbf{X}_{r^{(j)}}) > 0, 1 \leq j \leq n$. It follows from Theorem 2.2 (only if part) that

$$Rank_p(\sum_{i \in S} x_i) \geq \sum_{i \in S} r_i^{(j)} \text{ for all } S \subset \{1, 2, ..., n\}; 1 \leq j \leq n.$$

Therefore

$$Rank_p(\sum_{i \in S} x_i) \geq \sum_{i \in S} \sum_{1 \leq j \leq n} a_j r_i^{(j)} = \sum_{i \in S} r_i^{(j)}, S \subset \{1, 2, ..., n\}.$$

Using the "if" part of Theorem 2.2 we get that $M_p(\mathbf{X}_{r^{(0)}}) > 0$ and thus $r^{(0)} \in supp(q)$. ∎

Corollary 2.5: *Let $q(x), x \in R^n$ be S-hyperbolic polynomial of degree n . Then the following conditions are equivalent*

1. *$e \in CO(supp(q))$.*
2. *$e \in supp(q)$, i.e. $\frac{\partial^n}{\partial \alpha_1 ... \partial \alpha_n} q(x) > 0$.*
3. *$Cap(p) =: \inf_{\alpha_i > 0, \prod_{1 \leq i \leq n} \alpha_i = 1} q(\alpha_1, ..., \alpha_n) > 0.$*
4. *For all $\epsilon > 0$ there exists a vector $(\alpha_1, ..., \alpha_n)$ with positive entries such that the following inequality holds :*
$$\sum_{1 \leq i \leq n} |\frac{\alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1, ..., \alpha_n)}{q(\alpha_1, ..., \alpha_n)} - 1|^2 \leq \epsilon. \tag{16}$$
5. *There exists a vector $(\alpha_1, ..., \alpha_n)$ with positive entries such that the following inequality holds :*
$$\sum_{1 \leq i \leq n} |\frac{\alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1, ..., \alpha_n)}{q(\alpha_1, ..., \alpha_n)} - 1|^2 \leq \frac{1}{n}. \tag{17}$$
6. *For all subsets $S \subset \{1, 2, ..., n\}$ the following inequality holds :*
$$\sum_{i \in S} r_i \geq |S| \text{ for all } (r_1, ..., r_n) \in supp(q). \tag{18}$$

(We sketch a proof in Appendix **C** .)

The following result , which we prove in Appendix **B** , is a "polynomial" generalization of Lemma 4.2 in [20] .

Proposition 2.6: *The condition (17) implies the condition (18) for all homogeneous polynomial $q(x), x \in R^n$ of degree n in n real variables with nonnegative coefficients .*

2.1 Doubly-stochastic polynomials

Inequalities (16), (17) above suggest the following definition .

Definition 2.7: A homogeneous polynomial $q(x_1, ..., x_n)$ of degree n in n variables is called doubly-stochastic if its coefficients are nonnegative real numbers and $\frac{\partial}{\partial x_i} q(1, 1, ..., 1) = 1$ for all $1 \leq i \leq n$. The doubly-stochastic defect of the polynomial q is defined as $DS(q) = \sum_{1 \leq i \leq n} (\frac{\partial}{\partial x_i} q(1, 1, ..., 1) - 1)^2$ ∎

Lemma 2.8:

1. *A homogeneous polynomial $q(x_1, ..., x_n)$ of degree n in n variables with nonnegative real coefficients is doubly-stochastic iff $q(1, 1, ..., 1) = 1$ and $q(x_1, ..., x_n) \geq \prod_{1 \leq i \leq n} x_i$ for all real vectors $(x_1, ..., x_n) \in R^n$ with positive coordinates (in other words if $q(1, 1, ..., 1) = 1$ and $Cap(q) = 1$).*
2. *A homogeneous polynomial $q(x_1, ..., x_n)$ of degree n in n variables is P-hyperbolic and doubly-stochastic iff $q(1, 1, ..., 1) = 1$ and $|q(z_1, ..., z_n)| \geq \prod_{1 \leq i \leq n} Re(z_i)$ for all complex vectors vectors $(z_1, ..., z_n) \in C^n$ with positive real parts .*
3. *If a sequence q_i of homogeneous polynomials of degree n in n variables with nonnegative real coefficients converges to a doubly-stochastic polynomial then $\lim_{i \to \infty} Cap(q_i) = 1$.*
4. *The capacity $Cap(q)$ is a continuous functional (but not even Lipshitz) on a convex closed cone of homogeneous polynomials of degree n in n variables with nonnegative real coefficients .*
5. *If q is homogeneous polynomial of degree n in n variables with nonnegative real coefficients then $Cap(q) > 0$ iff the exists a sequence $X_j = (x_{1,j}, ..., x_{n,j})$ of vectors with positive real coordinates such that*
$$\sum_{1 \leq i \leq n} |\frac{x_i \frac{\partial}{\partial x_i} q(x_{1,j}, ..., x_{n,j})}{q(x_{1,j}, ..., x_{n,j})} - 1|^2 \to 0$$
And in this case $Cap(q) = \lim_{j \to \infty} \frac{q(X_j)}{\prod_{1 \leq i \leq n} x_{i,j}}$.

Example 2.9: A multilinear polynomial
$q(x_1, ..., x_n) = \prod_{1 \leq i \leq n} \sum_{1 \leq j \leq n} a(i,j) x_j$ is doubly-stochastic and P-hyperbolic iff the square matrix $B = \{ \frac{a(i,j)}{\sum_{1 \leq k \leq n} a(i,k)} : 1 \leq i, j \leq n \}$ is doubly stochastic in the standard meaning . ▮

The next theorem is another Corollary of Theorem 2.2 ; its main point is in introducing a hyperbolic analog of Van der Waerden conjecture .

Theorem 2.10:

1. *Consider the set $PHDS(n)$ of all P-hyperbolic doubly-stochastic homogeneous polynomials $q(x_1, ..., x_n)$ of degree n in n variables. The set $PHDS(n)$ is a compact and the following inequality holds*

$$\inf_{q \in PHDS(n)} \frac{\partial^n}{\partial x_1 ... \partial x_n} q(x_1, ..., x_n) =$$
$$\min_{q \in PHDS(n)} \frac{\partial^n}{\partial x_1 ... \partial x_n} q(x_1, ..., x_n) =: VdW(n) > 0$$

2. *For any P-hyperbolic homogeneous polynomials $q(x_1, ..., x_n)$ of degree n in n variables the following inequalities hold*

$$VdW(n) \leq \frac{\frac{\partial^n}{\partial x_1 ... \partial x_n} q(x_1, ..., x_n)}{Cap(q)} \leq 1.$$

Conjecture 2.11: Hyperbolic Van der Waerden conjecture

$$VdW(n) = \frac{n!}{n^n}?$$

▮

The next result , a direct corollary of Lemma 2.10 in [19] , is a generalization of Proposition 4.2 in [22].

Lemma 2.12: *Let q be P-hyperbolic homogeneous polynomial of degree n in n variables . Then the following inequalities hold for all vectors $(x_1, ..., x_n)$ with positive real coordinates.*

1.
$$Cap(q) \leq \frac{q(x_1, ..., x_n)}{\prod_{1 \leq i \leq n} x_i} \prod_{1 \leq i \leq n} x_i \frac{\frac{\partial}{\partial x_i} q(x_1, ..., x_n)}{q(x_1, ..., x_n)} \tag{19}$$

2. *If* $\log(\frac{q(x_1, ..., x_n)}{\prod_{1 \leq i \leq n} x_i}) - \log(Cap(q)) \leq \frac{\epsilon}{10}$ *with* $0 \leq \epsilon \leq 1$ *then*

$$\sum_{1 \leq i \leq n} |\frac{x_i \frac{\partial}{\partial x_i} q(x_1, ..., x_n)}{q(x_1, ..., x_n)} - 1|^2 \leq \epsilon \tag{20}$$

Example 2.13: Consider the following homogeneous polynomial of degree n in n variables : $p(x_1, ..., x_n) = \sum_{1 \leq i \leq n} x_i^n$. Then $Cap(p) = n$ and

$$\frac{p(x_1,...,x_n)}{\prod_{1 \leq i \leq n} x_i} \prod_{1 \leq i \leq n} x_i \frac{\frac{\partial}{\partial x_i} p(x_1,...,x_n)}{p(x_1,...,x_n)} =$$

$$= n^n \left(\frac{\prod_{1 \leq i \leq n} x_i}{\sum_{1 \leq i \leq n} x_i^n}\right)^{n-1} \leq n = Cap(p).$$

The moral of this example is that the inequality (19) does not hold for all homogeneous polynomials with nonnegative coefficients , this inequality is a nontrivial necessary condition for the P-hyperbolicity . It is interesting to notice that the inequality (19) implies the determinantal Hadamard inequality . ∎

Remark 2.14: Let

$$p(x_1, ..., x_n) = \sum_{(r_1,...,r_n) \in I_{n,n}} a_{(r_1,...,r_n)} \prod_{1 \leq i \leq n} x_i^{r_i}$$

be homogeneous polynomial of degree n in n real variables. Assume that its coefficients are nonnegative and sum to one , i.e. that $p(1, 1, ..., 1) = 1$. Associate with this polynomial a random integer vector

$$Z_p = (z_1, ..., z_n) \in I_{n,n} : Prob\{Z_p = (r_1, ..., r_n)\} = a_{(r_1,...,r_n)}.$$

Then

$$E(Z_p) = \left(\frac{\partial}{\partial x_1} p(1, 1, ..., 1), ..., \frac{\partial}{\partial x_n} p(1, 1, ..., 1)\right).$$

Therefore p is doubly-stochastic iff $E(Z_p) = (1, 1, ..., 1)$; there exists a doubly-stochastic polynomial q such that $supp(q) \in supp(p)$ iff $(1, 1, ..., 1) \in CO(supp(p))$.
It follows from Corollary 2.5 that if p is doubly-stochastic and S-hyperbolic then $Prob\{Z_p = E(Z_p)\} > 0$. And the hyperbolic van der Waerden conjecture can be reformulated as :
If p is doubly-stochastic and P-hyperbolic then
$Prob\{||Z_p = E(Z_p)|| < \sqrt{2}\} \geq \frac{n!}{n^n}$.
Perhaps some kind of the measure concetration is present here ? ∎

Remark 2.15: The problem to find out a positive real solution of the inequality (20) is a far reaching generalization of scaling of matrices with nonnegative entries (the corresponding polynomials are multilinear)[20] ,[23] and scaling of tuples of PSD matrices (the corresponding polynomials are determinantal) [21] , [22] . Part 2 of Lemma 2.12 allows to generalize results of [23] , [21] , [22] to P-hyperbolic polynomials , even in the black-box setting . Can it be done for all homogeneous polynomials with , say , integer nonnegative coefficients ? ∎

3 The ellipsoid algorithm

Consider a homogeneous polynomial $q(x), x \in R^n$ of degree n in n real variables with nonnegative integer coefficients . Associate with such q the following convex functional

$$f(y_1, ..., y_n) = \log(q(e^{y_1}, e^{y_2}, ..., e^{y_n})).$$

Proposition 3.1: *The following conditions are equivalent*

1. $e = (1, 1, .., 1) \in CO(supp(q))$.
2. $\inf_{y_1+...+y_n=0} f(y_1, ..., y_n) \geq 0$.
3. *If* $e = (1, 1, .., 1) \notin CO(supp(q))$ *then* $\inf_{y_1+...+y_n=0} f(y_1, ..., y_n) = -\infty$.
   Let $dist(e, CO(supp(q))) = \Delta^{-1} > 0$ *and* $Q = \log(q(e))$. *Define* $\gamma = (Q+1)\Delta$. *Then*

$$\inf_{y_1+...+y_n=0,(|y_1|^2+...+|y_n|^2)^{\frac{1}{2}} \leq \gamma} f(y_1, ..., y_n) =$$
$$\min_{y_1+...+y_n=0,|y_1|^2+...+|y_n|^2 \leq \gamma} f(y_1, ..., y_n) \leq -1.$$

Proof: Our proof is a straigthforward application of the concavity of the logarithm on the positive semi-axis and of the Hanh-Banach separation theorem . It will be included in the full version . ∎

Proposition 3.1 suggests the following natural approach to solve Problem 2 , i.e. to decide whether $e = (1, 1, .., 1) \in CO(supp(q))$ or not :
find $\min_{y_1+...+y_n=0,|y_1|^2+...+|y_n|^2 \leq \gamma} f(y_1, ..., y_n)$ with absolute accuracy $\frac{1}{3}$. If the resulting value is greater than or equal $-\frac{1}{3}$ then $e = (1, 1, .., 1) \in CO(supp(q))$; if the resulting value is less than or equal $-\frac{2}{3}$ then $e = (1, 1, .., 1) \notin CO(supp(q))$. And , of course , it is natural to use the ellipsoid method . Our main tool is the following property of the ellipsoid algorithm [32]: For a prescribed accuracy $\delta > 0$, it finds a δ-minimizer of a differentiable convex function f in a ball B, that is a point $x_\delta \in B$ with $f(x_\delta) \leq \min_B f + \delta$, in no more than

$$O\left(n^2 \ln\left(\frac{2\delta + \mathrm{Var}_B(f)}{\delta}\right)\right), \qquad (\mathrm{Var}_B(f) = \max_B f - \min_B f)$$

iterations. Each iteration requires a single computation of the value and of the gradient of f at a given point, plus $O(n^2)$ elementary operations to run the algorithm itself. In our case, this is easily seen to cost at most $O(n^2)$ oracle calls and $O(n)$ elementary arithmetic operations .
We have the $n-1$-dimensional ball $B_\gamma = \{(y_1, ..., y_n) : y_1 + ... + y_n = 0, |y_1|^2 + ... + |y_n|^2 \leq \gamma\}$.
A straigthforward computations show that

$$Var_B(f) \leq \log(q(1, 1, .., 1)e^{\gamma n}) - \log(q(1, 1, .., 1)e^{-\gamma n}) \leq 2\gamma n,$$

giving that $O(n^2(\ln(n) + \ln(\gamma))$ iterations of the ellipsoid method needed to solve Problem 2 , it amounts to $O(n^4(\ln(n) + \ln(\gamma))$ oracle calls . The quantity $O(n^4(\ln(n) + \ln(\gamma))$ is polynomial in n even if γ is exponentially large ($dist(e, CO(supp(q)))$ is exponentially small). The problem is that if γ is exponentially large (which can happen) then we need to call oracles on inputs with exponential bit-size .
Putting things together , we get the following conclusion :
If it is promised that either $e = (1, 1, .., 1) \in CO(supp(q))$ *or* $dist(e, CO(supp(q))) \geq poly(n)^{-1}$ *for some fixed polynomial* $poly(n)$ *then Problem 2 can be solved by a deterministic polynomial-time oracle algorithm based on the ellipsoid method .*
And at this point we can say nothing about Problem 1 , i.e. deciding whether $e = (1, 1, .., 1) \in$

$supp(q)$ or not . Corollary 2.5 says that if q is S-hyperbolic polynomial then Problem 1 and Problem 2 are equivalent ; moreover if $e = (1, 1, .., 1) \notin supp(q)$ then here exists nonempty $S \subset \{1, 2, ..., n\}$ such that

$$\sum_{1 \le i \le n} r_i s_i < \sum_{1 \le i \le n} s_i = |S| \text{ for all}(r_1, ..., r_n) \in supp(q), \tag{21}$$

, where $(s_1, ..., s_n)$ is a characteristic function of the subset S , i.e. $s_i = 1$ if $i \in S$, and $s_i = 0$ otherwise .
Notice that if (21) holds then the distance $dist(e, CO(supp(q)))$ from the vector $e = (1, ..., 1)$ to the Newton polytope $CO(supp(q))$ is at least $\sqrt{\frac{n}{|S|(n-|S|)}} \ge \frac{2}{\sqrt{n}}$. Thus we have the next theorem .

Theorem 3.2: *Problem 1 and Problem 2 are equivalent for S-hyperbolic polynomials .*
There exists a deterministic polynomial-time oracle algorithm solving Problem 1 for a given S-hyperbolic polynomial $q(\alpha_1, ..., \alpha_n)$ with integer coefficients .
It requires $O(n^4(\ln(n) + \ln(\ln(q(1, 1, ..., 1)))$ oracle calls and it bit-wise complexity (which is roughly the radius of the ball B_γ) is $O(n^{\frac{1}{2}} \ln(q(1, 1, ..., 1)))$.

4 Hyperbolic Sinkhorn scaling

We will discuss briefly in this section another method , which is essentially a large step version of gradient descent .

Definition 4.1: Consider an e-nonnegative tuple $\mathbf{X} = (x_1, .., x_n)$ such that the sum of its components $S(\mathbf{X}) = d = \sum_{1 \le i \le k} x_i$ is e-positive. Define $tr_d(x)$ as a sum of roots of the univariate polynomial equation $p(x - td) = 0$.
Define the following map (Hyperbolic Sinkhorn Scaling) acting on such tuples:

$$HS(\mathbf{X}) = \mathbf{Y} = (\frac{x_1}{tr_d(x_1)}, ..., \frac{x_n}{tr_d(x_n)})$$

Hyperbolic Sinkhorn Iteration (**HSI**) is the following recursive procedure:

$$\mathbf{X}_{j+1} = HS(\mathbf{X}_j), j \ge 0,\ \mathbf{X}_0 \text{ is an } e\text{-nonnegative tuple with } \sum_{1 \le i \le k} x_i \in C_e .$$

We also define the doubly-stochastic defect of e-nonnegative tuples with e-positive sums as

$$DS(\mathbf{X}) = \sum_{1 \le i \le k} (tr_d(x_i) - 1)^2; \sum_{1 \le i \le k} x_i = d \in C_e$$

∎

We can define the map $HS(.)$ directly in terms of the P-hyperbolic polynomial

$$Q(\alpha_1, ..., \alpha_n) = P_{x_1,..,x_n}(\alpha_1, ..., \alpha_n) = p(\sum_{1 \le i \le n} \alpha_i x_i).$$

Indeed, if $\sum_{1 \leq i \leq n} \alpha_i x_i = d \in C_e$ then

$$tr_d(\alpha_i x_i) = \frac{\alpha_i \frac{\partial}{\partial \alpha_i} Q(\alpha_1, ..., \alpha_n)}{Q(\alpha_1, ..., \alpha_n)} \tag{22}$$

This gives the following way to redefine the map $HS(\mathbf{X})$:

$$HS(\alpha_1, ..., \alpha_n) = (\frac{Q(\alpha_1, ..., \alpha_n)}{\frac{\partial}{\partial \alpha_1} Q(\alpha_1, ..., \alpha_n)}, ..., \frac{Q(\alpha_1, ..., \alpha_n)}{\frac{\partial}{\partial \alpha_n} Q(\alpha_1, ..., \alpha_n)}),$$

for $\alpha_i > 0, 1 \leq i \leq n$.
And correspondingly the doubly-stochastic defect of $(\alpha_1, ..., \alpha_n)$ is equal to

$$\sum_{1 \leq i \leq n} |\frac{\alpha_i \frac{\partial}{\partial \alpha_i} Q(\alpha_1, ..., \alpha_n)}{Q(\alpha_1, ..., \alpha_n)} - 1|^2,$$

the same as the left side of (17) . Notice that $\sum_{1 \leq i \leq n} tr_d(x_i) = n$ by the Euler's identity .

Example 4.2: Consider the following hyperbolic polynomial in n variables: $p(z_1, ..., z_n) = \prod_{1 \leq i \leq n} z_i$. It is e- hyperbolic for $e = (1, 1, ..., 1)$. And N_e is a nonnegative orthant, C_e is a positive orthant. An e-nonnegative tuple $\mathbf{X} = (x_1, .., x_n)$ can be represented by an $n \times n$ matrix $A_{\mathbf{X}}$ with nonnegative entries: the ith column of A is a vector $x_i \in R^n$. If $Z = (z_1, ..., z_n) \in R^n$ and $d = (d_1, ..., d_n) \in R^n; z_i > 0, 1 \leq i \leq n$, then $tr_d(Z) = \sum_{1 \leq i \leq n} \frac{z_i}{d_i}$. Recall that for a square matrix $A = \{a_{ij} : 1 \leq i, j \leq N\}$ row scaling is defined as

$$R(A) = \{\frac{a_{ij}}{\sum_j a_{ij}}\},$$

column scaling as $C(A) = \{\frac{a_{ij}}{\sum_i a_{ij}}\}$ assuming that all denominators are nonzero. The iterative process $...CRCR(A)$ is called *Sinkhorn's iterative scaling* (SI). In terms of the matrix $A_{\mathbf{X}}$ the map $HS(\mathbf{X})$ can be realized as follows:

$$A_{HS(\mathbf{X})} = C(R(A_{\mathbf{X}}))$$

So, the map $HS(\mathbf{X})$ is indeed a (rather far-reaching) generalization of Sinkhorn's scaling. Other generalizations (not all hyperbolic) can be found in [25], [3], [2]. ∎

Lemma 2.10 from [19] allows to use (**HSI**) to solve Problem 1 for P-hyperbolic polynomials q in the same way as it was done for the perfect matching problem in [25] , [20] ; and for the Edmonds' problem in [3] . The corresponding complexity is $O(n \log(q(e)))$ iterations of (**HSI**) , which can be done in $O(n^3 \log(q(e)))$ oracle calls . The algorithm works in the following way : *Run* $K = O(n \log(q(e)))$ *Hyperbolic Sinkhorn Iterations* $\mathbf{X}_{j+1} = HS(\mathbf{X}_j)$ *; if* $DS(\mathbf{X}_i) \leq \frac{1}{n}$ *for some* $i \leq K$ *then the p-mixed form* $M_p(\mathbf{X}_0) > 0$ *, and* $M_p(\mathbf{X}_0) = 0$ *otherwise .*

5 Half-Plane Property

The following definition is from [7].

Definition 5.1: A polynomial $P(z_1, ..., z_n)$ in n complex variables is said to have the "half-plane property" if $P(z_1, ..., z_n) \neq 0$ provided $Re(z_i) > 0$. ▮

In a control theory literature (see [36]) the same property is called *Wide sense stability* . And *Strict sense stability* means that
$P(z_1, ..., z_n) \neq 0$ provided $Re(z_i) \geq 0$.

The following simple fact shows that for homogeneous polynomials the "half-plane property" is , up to a single factor , the same as P-hyperbolicity .

Fact 5.2: A homogeneous polynomial $R(z_1, ..., z_n)$ has the "half-plane" property if and only if the exists real α such that the polynomial $e^{i\alpha}R(z_1, ..., z_n)$ is P-hyperbolic polynomial with real nonnegative coefficients . ▮

Proof:

1. Suppose that $R(z_1, ..., z_n) = e^{-i\alpha}Q(z_1, ..., z_n)$ where α is real and Q is P-hyperbolic. Then Q is $(1, 1, ...,)$-hyperbolic and all real vectors $(x_1, ..., x_n)$ with positive coordinates are $(1, 1, ...,)$-positive . Therefore Q is $(x_1, ..., x_n)$-hyperbolic for all real vectors $(x_1, ..., x_n) \in R^n_{++}$ with positive coordinates .It follows that $|R(x_1 + iy_1, ..., x_n + iy_n)| = |Q(x_1 + iy_1, ..., x_n + iy_n)| = |Q(x_1, ..., x_n)\prod_{1 \leq k \leq n}(1 + i\lambda_k)|$, where $(\lambda_1, ..., \lambda_n)$ are real roots of the real vector $(y_1, ..., y_n)$ in the direction $(x_1, ..., x_n)$.
   This gives the following inequality , which is equivalent to the "half-plane property" of R :
   $$|R(x_1 + iy_1, ..., x_n + iy_n)| \geq |R(x_1, ..., x_n)| = \\ = |Q(x_1, ..., x_n)| > 0 : \\ (x_1, ..., x_n) \in R^n_{++}, (y_1, ..., y_n) \in R^n \tag{23}$$
2. Suppose that $R(z_1, ..., z_n)$ has the "half-plane property" and consider the roots of the following polynomial equation in one complex variable : $P(x_1 - z, x_2 - z, ..., x_n - z) = 0$, where $(x_1, ..., x_n) \in R^n$ is a real vector , $z = x + iy \in C$. If the imaginery part $Im(z) = y$ is not zero then , using the homogeniuty , $R(i\frac{x-x_1}{y} + 1, ..., i\frac{x-x_n}{y} + 1) = 0$, which is impossible as R has the "half-plane property". Therefore all roots of $R(X - te) = 0$ are real for all real vectors $X \in R^n$ (here $e = (1, 1, ..., 1)$). In the same way all roots of $R(X - te) = 0$ are real positive numbers if $X \in R^n_{++}$. It follows that if $X \in R^n$ then $R(X) = R(e)\prod 1 \leq k \leq n\lambda_k(X)$, where $(\lambda_1, ..., \lambda_n)$ are (real) roots of the equation $R(X - te) = 0$. Thus the polynomial $(\frac{1}{R(e)})R$ takes real values on R^n and therefore its coefficients are real . In other words , the polynomial $(\frac{1}{R(e)})R$ is P-hyperbolic . If $R(1, 1, ..., 1) = e^{-i\alpha}|R(1, 1, ..., 1)|$ then the polynomial $e^{i\alpha}R$ is also P-hyperbolic .
   (Recall that the coefficients of any P-hyperbolic polynomial p are nonnegative for they are p-mixed forms of e-nonnegative tuples , and p-mixed forms of e-nonnegative tuples are nonnegative if $p(e) > 0$ [26].)

$\blacksquare$

We use this observation to show that Theorem 2.2 in this paper implies (and seriously strengthens) Theorem 7.2 in [7] , which is the main result of a very long recent paper [7] .

5.1 Submodularity and hyperbolicity

Let p be a P-hyperbolic polynomial of degree n in n variables . It follows from Theorem 2.2 that $r = (r_1, r_2, ..., r_n) \in supp(p)$ if and only if the following inequalities hold :

$$r(S) = \sum_{i \in S} r_i \leq R(S) = Rank_p(\sum_{i \in S} e_i); S \subset \{1, 2, ..., n\}.$$

Fact 5.3: The functional $R(S) = Rank_p(\sum_{i \in S} e_i)$ is normalized , i.e. $R(\emptyset) = 0$, and submodular , i.e. $R(A \cup B) \leq R(A) + R(B) - R(A \cap B) : A, B \subset \{1, 2, ..., n\}$. $\blacksquare$

Proof: Associate with two subsets $A, B \subset \{1, 2, ..., n\}$ the following three e-nonnegative vectors :

$x = \sum_{i \in A \setminus (A \cap B)} e_i, y = \sum_{i \in A \cap B} e_i, z = \sum_{i \in B \setminus (A \cap B)} e_i$.
We need to prove the inequality $Rank_p(x + y + z) \leq Rank_p(x) + Rank_p(z) - Rank_p(y)$. This inequality is obvious and well known for positive semidefinite matrices . The extension to e-nonnegative vectors respect to e-hyperbolic polynomial p is done in the same way as in the proof of Corollary A.3 : consider a hyperbolic in the direction $(1, 1, 1)$ polynomial

$$L(\alpha_1, \alpha_2, \alpha_3) = M_p(k, ..., k, e, ..., e), k = \alpha_1 x + \alpha_2 y + \alpha_3 z;$$

where the vectors x, y, z are e-nonnegative respect to hyperbolic polynomial p , and the tuple $(k, ..., k, e, ..., e)$ consists of $Rank_p(x + y + z)$ copies of k and $n - Rank_p(x + y + z)$ copies of e. After that apply Theorem 1.5 . $\blacksquare$

Corollary 5.4:

1. *A support $supp(p)$ of P-hyperbolic polynomial p is an intersection of the integral polymatroid $\{(r_1, ..., r_n) : r(S) = \sum_{i \in S} r_i \leq R(S) = Rank_p(\sum_{i \in S} e_i); S \subset \{1, 2, ..., n\}\}$ with the hyperplane $\{(r_1, ..., r_n) : \sum_{1 \leq i \leq n} r_i = n\}$.*

2. *A support $supp(R)$ of any polynomial R with the "half-plane" property is a jump system .*

Proof: (Consult [24] for a definition and some properties of jump systems and integral polymatroids) . This Corollary follows directly Theorem 2.2 , Fact 5.3 and Proposition (3.1) in [24] . $\blacksquare$
It is quite amazing how the two communities , "hyperbolic" and "half-plane" , were not aware about each other results for a long , long time . (Interestingly , two authors of [7] and one author of [27] were with the same department until very recently . Perhaps , one needs to be a dilettante to notice a bridge .)

6 Conclusion and Acknowledgments

Univariate polynomials with real roots appear quite often in modern combinatorics , especially in the context of integer polytopes . We discovered in this paper rather unexpected and very likely far-reaching connections between hyperbolic polynomials and many classical combinatorial and algorithmic problems . (The author taught about "On hyperbolic nature of perfect marriages" as a title of this paper , but with the current climate it could be understood in many ways .) There are still several open problems . The most interesting is Conjecture 2.11 in this paper , which is a generalization of the van der Waerden conjecture for permanents of doubly stochastic matrices and many others related questions .
For a hyperbolic in direction $(1, 1, .., 1)$ polynomial $Mul(y_1, ..., y_n) = y_1 y_2 ... y_n$ Conjecture 2.11 is equivalent to the famous van der Waerden conjecture for permanents of doubly stochastic matrices , proved in [15] , [16] . For a hyperbolic in direction I polynomial $\det(X)$, X is $n \times n$ hermitian matrix , it is equivalent to Bapat's conjecture [5] (it was also hinted in [15]) , proved by the author in [21] , [35] . It also holds for the Moore determinant $Det(M)(Y)$, Y is $n \times n$ quaternionic hermitian matrix , with the proof essentially the same as in [35] .
Another , equivalent form of "hyperbolic" (or "half-plane") van der Waerden conjecture can be formulated as follows :

Conjecture 6.1: Consider a homogeneous polynomial $p(z_1, ..., z_n)$ of degree n in n complex variables . Assume that this polynomial satisfies the property :

$|p(z_1, ..., z_n)| \geq \prod_{1 \leq i \leq n} Re(z_i)$ on the domain $\{(z_1, ..., z_n) : Re(z_i) \geq 0, 1 \leq i \leq n\}$.

Is it true that $|\frac{\partial^n}{\partial z_1 ... \partial z_n} p| \geq \frac{n!}{n^n}$? .
(Notice that Theorem 2.10 and Fact 5.2 imply that $\frac{\partial^n}{\partial z_1 ... \partial z_n} p \neq 0$.) ∎

It would be very interesting and enlighting to prove Conjecture 2.11 using methods of the theory of functions of many complex variables. Fact 5.2 , together with other results in this paper , makes a connection between the Complexity Theory and the theory of linear time-miltidimensional systems : all "hard" instances of Problem 1 are necessary unstable polynomials.
Another interesting conjecture is related to the majorization :

Conjecture 6.2: Consider the doubly-stochastic and P-hyperbolic homogeneous polynomial $p(x_1, ..., x_n)$ of degree n in n real variables .
Let $\Lambda(X) \in R^n$ be a real n-dimensional vector , whose coordinates are the roots of the equation $p(X - te) = 0$, where $X \in R^n$ and e is the vector of all ones . Then there exists a $n \times n$ doubly stochastic matrix A such that $\Lambda(X) = AX$.

(Some partial and related results in this direction can be found in [19] ; this conjecture is true for determinantal polynomials .) ∎

A natural extension of **Problem 1** is for P-hyperbolic polynomials to approximate $\frac{\partial^n}{\partial x_1 ... \partial x_n} p(x_1, ..., x_n)$ within a multiplicative factor using deterministic (or randomized) polynomial-time oracle algorithms . It is not clear to the author whether known recent randomized algorithms for $(1+\epsilon)$ approximation of the permanent $Per(B)$ of entry-wise nonnegative matrix B can be done in the "oracle fashion" , i.e. using only some outputs of the multilinear polynomial

$$q(x_1, ..., x_n) = \prod_{1 \leq i \leq n} \sum_{1 \leq j \leq n} B(i,j) x_j .$$

If hyperbolic van der Waerden conjecture is true then the technique in this paper , similarly to [20] and [21] , [22] , would produce a deterministic polynomial-time oracle algorithm with $\frac{n^n}{n!}$ multiplicative factor .

The technique developed in this paper can be applied to other "noble" desicion problems . For instance , checking factorizability of P-hyperbolic polynomials can be also done in deterministic oracle polynomial time . The factorizability is closely related to the hyperbolic generalization of the indecomposability of matrix tuples [22].
This paper is probably the first one which uses Theorem 1.5 in the combinatorial context . We expect many more such applications of Theorem 1.5 . This (very nontrivial) theorem , when in good hands , is a powerful tool allowing reasonably simple and short proofs .

I would like to acknowledge a great influence of amazingly clear paper [26] . It is my pleasure to thank Adrian Lewis for numerous as e-mail as well phone communications. Many thanks to the fantastic library of Los Alamos National Laboratory and Google.

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A Proof of the (main) Theorem 2.2

Before proving Theorem 2.2 , we will recall some basic properties of p-mixed forms and prove a few auxillary results . The following fact was proved in [26]

Fact A.1: Consider a homogeneous polynomial $p(x), x \in R^m$ of degree n in m real variables which is hyperbolic in the direction e. Then the following properties hold .

1. The p-mixed form $M_p(x_1, .., x_n)$ is linear in each $x_i, 1 \leq i \leq n$.
2. If $x_1, x_2, .., x_{n-1}$ are e-nonnegative then the linear functional $l(x) = M_p(x_1, .., x_{n-1}, x)$ is nonnegative on the closed cone N_e of e-nonnegative vectors .
3. If the tuples $(x_1, .., x_n), (y_1, .., y_n), (x_1 - y_1, .., x_n - y_n)$ are e-nonnegative then
$$0 \leq M_p(y_1, .., y_n) \leq M_p(x_1, .., x_n).$$
4. Fix e-positive vector d and consider the following homogeneous polynomial $p_d(x), x \in R^m$ of degree $n-1$ in m real variables : $p_d(x) =: M_p(x, x, ..., x, d)$. Then p_d is hyperbolic in any e-positive direction $v \in C_e(p)$. If $g \in C_e(p)$ (e-positive respect to the polynomial p) then also $q \in C_v(p_d)$ for all $v \in C_e(p)$.

∎

The next fact is well known .

Fact A.2: Consider a sequence of univariate polynomials of the same degree n : $P_k(t) = \sum_{0 \leq i \leq n} a_{i,k} t^i$. suppose that $\lim_{k \to \infty} a_{i,k} = a_i, 0 \leq i \leq n$ and $a_n \neq 0$.
Define $P(t) = \sum_{0 \leq i \leq n} a_i t^i$. Then roots of P_k converge to roots of P . In particular if roots of all polynomials P_k are real then also roots of P are real ; if roots of all polynomials P_k are real nonnegative numbers then also also roots of P are real nonnegative numbers . ∎

The following corollary of Theorem 1.5 plays crucial role in our proof of Theorem 2.2 .

Corollary A.3:

1. *Consider a homogeneous polynomial $p(x), x \in R^m$ of degree n in m real variables which is hyperbolic in the direction e. Let x_1, x_2, x_3 be three e-nonnegative vectors and $d = x_1 + x_2 + x_3$ is e-positive . Assume wlog that $p(x_1 + x_2 + x_3) = 1$. Then there exists three symmetric positive semidefinite matrices A, B, C such that $p(a_1x_1 + a_2x_2 + a_3x_3) = \det(a_1A + a_2B + a_3C0$ for all real a_1, a_2, a_3. Additionally , the roots of $a_1x_1 + a_2x_2 + a_3x_3$ in the direction d , i.e. the roots of the equation $p(a_1x_1 + a_2x_2 + a_3x_3 - td) = 0$, coincide with the eigenvalues of $a_1A + a_2B + a_3C$.*
2. *Theorem 2.2 is true for e-nonnegative tuples $(\mathbf{X}) = (x_1, ...x_n), x_i \in R^m$ consisting of at most three distinct components , i.e the cardinality of the set $\{x_1, ...x_n\}$ is at most three .*

Proof:

1. Consider the following homogeneous polynomial $L(b_1, b_2, b_3) = p(b_1x_1 + b_2x_2 + b_3(x_1 + x_2+x_3))$ of degree n in 3 real variables . It follows from Theorem 1.5 that there exists two real symmetric matrices A and B such that $L(b_1, b_2, b_3) = \det(b_1A+b_2B+b_3I)$. It follows that they both positive semidefinite , and $C = I - A - B$ is also positive semidefinite . Take a real linear combination $z = a_1x_1 + a_2x_2 + a_3x_3$. Then

$$
\begin{gathered}
p(z - t(x_1 + x_2 + x_3)) = \\
\det((a_1 - a_3)A + (a_2 - a_3)B + a_3I - tI) = \\
= \det(a_1A + a_2B + a_3C - tI).
\end{gathered}
$$

This proves that $p(a_1x_1 + a_2x_2 + a_3x_3) = \det(a_1A + a_2B + a_3C)$ for all real a_1, a_2, a_3 by putting $t = 0$. And it also proves the "eigenvalues " statement .

2. Consider e-nonnegative tuple $(\mathbf{X})$ consisting of r_i copies of x_i , $1 \le i \le 3$; $r_1+r_2+r_3 = n$. Assume that $d = x_1 + x_2 + x_3$ is e-positive (if it is not then $M_p(\mathbf{X}) = 0$ by a simple argument based on the monotonicity of p-mixed forms). It follows from the polarization formula (10) that

$$
M_p(\mathbf{X}) = \sum_{1 \le i \le k < \infty} d_i p(t_{1,i}x_1 + t_{2,i}x_2 + t_{3,i}x_3),
$$

and this formula is universal , i.e. holds for all homogeneous polynomial of degree n , in particular for $\det(X)$, X is $n \times n$ symmetric matrix . Therefore , using the first part of this Corollary we get that the p-mixed form $M_p(\mathbf{X}) = D(\mathbf{A})$, where the matrix tuple $\mathbf{A}$ consists of r_1 copies of A , r_2 copies of B and r_3 copies of C and $D(\mathbf{A})$ is the mixed discriminant . Using Rado theorem for mixed discriminants we get that $D(\mathbf{A}) > 0$ iff

$$
Rank(\sum_{i \in S} A_i) \ge \sum_{i \in S} r_i \quad \text{for all} \quad S \subset \{1, 2, 3\}.
$$

But from the first part we get that $Rank(\sum_{i \in S} A_i)$ is equal to p-rank $Rank_p(\sum_{i \in S} x_i)$ of $\sum_{i \in S} x_i$ for all $S \subset \{1, 2, 3\}$.

∎

Proposition A.4: *Consider similarly to part 4 of Fact A.1 the polynomial* $p_d(x) =: M_p(x, x, ..., x, d)$ *where* d *is e-nonnegative and* $Rank_p(d) \ge 1$ *. Then* p_d *is hyperbolic in any direction* $z \in N_e(p)$ *which is e-nonnegative and satisfies the following inequalities :*

$$
Rank_p(z) \ge n - 1; \;\; Rank_p(z + d) = n. \tag{24}
$$

Also , if $y \in N_e(p)$ *is e-nonnegative then also* $y \in N_z(p_d)$ *, i.e. is z-nonnegative respect to the polynomial* p_d*.*

Proof: Let $z \in N_e(p)$ be e-nonnegative vector satisfying (24) . Consider univariate polynomial $P(t) = M_p(tz+x, tz+x, ..., tz+x, d)$. Then $P(t) = \sum_{0 \le i \le n-1} a_i t^i$ and $a_{n-1} = M_p(z, z, ..., z, d)$. It follows from Corollary A.3 that $a_{n-1} > 0$. Consider now a sequence of univariate polynomials

$P_k(t) = M_p(tz_k + x, tz_k + x, ..., tz_k + x, d_k)$. Where z_k, d_k are e-positive and $\lim_{k\to\infty} z_k = z,\ \lim_{k\to\infty} d_k = d$. Then the coefficients of polynomials P_k converge to the coefficients of the polynomial P . It follows from part 4 of Fact A.1 that the roots of P_k are real . Since $a_{n-1} > 0$ hence using Fact A.2 we get that the roots of P are also real . This exactly means that the polynomial p_d is hyperbolic in direction z . The d-nonnegativity statement follows from the nonnegativity part of Fact A.2 . ∎

We are ready now to present our proof of Theorem 2.2 . The proof is by induction in the degree n . The main trick which we used is that to justify the induction , i.e. that if the generalized Rado conditions hold for hyperbolic polynomial p of degree n then the generalized Rado conditions hold for some auxillary hyperbolic polynomial p_d of degree $n-1$, we need to prove Theorem 2.2 for tuples consisting of at most three distinct components . And this particular case follows from the classical Rado theorem via Theorem 1.5 and Corollary A.3 .

Proof: (**Proof of Theorem 2.2**) .

The "only if" part is simple . Indeed supposed that there exists a subset $S \subset \{1, 2, ..., n\}$ such that $Rank_p(\sum_{i\in S} x_i) < |S|$, i.e. using the identities (14) $M_p(k, k, ...k, d, .., d) = 0$, where $k = \sum_{i\in S} x_i$, $d \in C_e(p)$ is e-positive and the n-tuple $(k, k, ...k, d, .., d)$ consists of $|S|$ copies of $k = \sum_{i\in S} x_i$. Let d be any e-positive positive vector such that $d - x_i$ is e-nonnegative , $1 \le i \le n$. Using the monotonicity of p-mixed forms we get that

$$M_p(x_1, ..., x_n) \le M_p(k, k, ...k, d, .., d) = 0.$$

Our proof of the "if" part is by induction in the degree n . Suppose that the generalized Rado conditions (15) hold . Then at least $Rank_p(x_n) \ge 1$. Consider the following homogeneous polynomial of degree $n-1$:

$$p_d(x) = M_p(x, x, ..., x, d),\ \ d = x_n.$$

We get from Proposition A.4 the following assertion :
The polynomial $p_d(x)$ is hyperbolic in direction $z = \sum_{1\le i\le n-1} x_i$ and the vectors $x_i \in N_z(p_d), 1 \le i \le n-1$, i.e. are z-nonnegative respect to the polynomial p_d.
Indeed , it follows from the generalized Rado conditions (15) that $Rank_p(z) \ge n-1$ and $Rank_p(z + d) = Rank_p(\sum_{1\le i\le n} x_i) = n$.

Next we show that the $n-1$-tuple $\mathbf{Y} = (x_1, ..., x_{n-1})$ satisfies the generalized Rado conditions for z-hyperbolic polynomial p_d of degree $n-1$:

$$Rank_{p_d}(\sum_{i\in S} x_i) \ge |S| \ \text{ for all } \ S \subset \{1, 2, ..., n-1\}.$$

Or equivalently (see formulas (14)) , that

$$M_p(k, .., k, z, ..., z, d) > 0; k = \textstyle\sum_{i\in S} x_i, \tag{25}$$

$$z = \textstyle\sum_{1\le i\le n-1} x_i, d = x_n, S \subset \{1, ..., n-1\}, \tag{26}$$

where the n-tuple $\mathbf{T} = (k, .., k, z, ..., z, d)$ consists of $|S|$ copies of k , $n-1-|S|$ copies of z and one copy of d .

It is easy to see that the generalized Rado conditions for the n-tuple $\mathbf{T}$ are implied by the generalized Rado conditions for the original n-tuple $\mathbf{X} = (x_1, ..., x_{n-1}, x_n)$. Since the n-tuple $(k, .., k, z, ..., z, d)$ consists of at most three distinct components hence we can apply part 2 of Corollary A.3 . Therefore we get that indeed
$M_p(k, .., k, z, ..., z, d) > 0$ and hence the following inequalities hold :

$$Rank_{p_d}(\sum_{i \in S} x_i) \geq |S| \ \text{ for all } \ S \subset \{1, 2, ..., n-1\}. \tag{27}$$

Thus , by induction in the degree , we get that p_d-mixed form $M_{p_d}(x_1, ..., x_{n-1}) > 0$:
the polynomial p_d of degree $n-1$ in m real variables is z-hyperbolic . But

$$M_{p_d}(x_1, ..., x_{n-1}) = \tfrac{\partial^{n-1}}{\partial \alpha_1 ... \partial \alpha_{n-1}} p_d(\textstyle\sum_{1 \leq i \leq n-1} \alpha_i x_i) =$$
$$= \tfrac{\partial^{n-1}}{\partial \alpha_1 ... \partial \alpha_{n-1}} M_p(\textstyle\sum_{1 \leq i \leq n-1} \alpha_i x_i, ..., \sum_{1 \leq i \leq n-1} \alpha_i x_i, x_n) = (n-1)! M_p(x_1, ..., x_n).$$

We conclude that if Theorem 2.2 is true for $n-1$ then it is also true for n , and the case "$n = 1$" is trivially true . ∎

Remark A.5: Consider a mixed discriminant $D(\mathbf{A})$, where $\mathbf{A} = (A_1, ..., A_n)$ is a n-tuple of positive semidefinite $n \times n$ hermitian matrices , i.e. $A_i \succeq 0$. Recall that in this case $D(\mathbf{A}) \geq 0$; and $D(\mathbf{A}) > 0$ if and only if there exists n linearly independent vectors $v_1, ..., v_n$ such that $v_i \in Im(A_i), 1 \leq i \leq n$.
In the proof of Theorem 2.2 we encountered the following tuple of positive semidefinite matrices :
$\mathbf{A} = (A, ..., A, B, ..., B, C)$ consisting of l copies of A , m copies of B and one copy of C . Moreover , this tuple is even more special . I.e. $B - A \succeq 0$, $Rank(B) \geq n-1$, $Rank(A) \geq l$, $rank(C) \geq 1$,$Rank(A + C) \geq l + 1$ and $Rank(B + C) = n$.
For such tuples the Rado theorem has very elementary proof , which we sketch below .

There are two cases . First case is when $Rank(B) = n$, it is simple and left to the reader . Second case is when $Rank(B) = n - 1$.
This is how in this case we can choose vectors $v_n \in Im(C); v_1, ..., v_l \in Im(A); v_{l+1}, ..., v_{n-1} \in Im(B)$ in such a way that $(v_1, ..., v_n)$ is a basis : first choose nonzero $v_n \in Im(C)$ which does not belong to $Im(B)$, second choose any l linearly independent vectors $v_1, ..., v_l \in Im(A$, third choose any $n - l - 1$ linearly independent vectors in $Im(B) \cap L(v_1, ..., v_l)^{\perp}$. $(L(v_1, ..., v_l)^{\perp}$ is a linear subspacespace orthogonal to the linear subspace $L(v_1, ..., v_l)$ which is spanned by $(v_1, ..., v_l)$.)
∎

B Proof of Proposition 2.6

Proof: Assume wlog that $q(\alpha_1, ..., \alpha_n) = 1$. It follows from the Euler's identity that

$$\sum_{1 \leq i \leq n} \alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1, ..., \alpha_n) = n.$$

Let $q(\alpha_1,...,\alpha_n) = \sum_{(r_1,...,r_n)\in supp(q)} a_{(r_1,...,r_n)} \prod_{1\leq i\leq n} \alpha_i^{r_i}$.
Define the following nonnegative real numbers :

$$b_{(r_1,...,r_n)} = a_{(r_1,...,r_n)} \prod_{1\leq i\leq n} \alpha_i^{r_i}, (r_1,...,r_n) \in supp(q).$$

Then $\alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1,...,\alpha_n) = \sum_{(r_1,...,r_n)\in supp(q)} r_i b_{(r_1,...,r_n)}$.

Suppose that for some subset $S \subset \{1,2,...,n\}, 1 \leq |S| < n$ we have the inequality $\sum_{i\in S} r_i < |S|$ for all $(r_1,...,r_n) \in supp(q)$. Then $\sum_{i\in S} \alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1,...,\alpha_n) \leq |S|-1$. But the condition (17) says that $\alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1,...,\alpha_n) = 1+\delta_i$ and $\sum_{1\leq i\leq n} |\delta_i|^2 \leq \frac{1}{n}$. By the Cauchy-Schwarz inequality , $\sum_{i\in S} |\delta_i| \leq \sqrt{\frac{|S|}{n}} < 1$. Therefore ,

$$\sum_{i\in S} \alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1,...,\alpha_n) \geq |S| - \sum_{i\in S} |\delta_i| > |S| - 1.$$

The last inequality gives a contradiction . ∎

C A sketch of a proof of Corollary 2.5

Proof: By Theorem 2.2 the conditions (1) and (2) are equivalent . (2) implies (3) for any homogeneous polynomial with nonnegative coefficients .
Let $\alpha_i = e^{y_i}, 1 \leq i \leq n; \sum_{1\leq i\leq n} y_i = 0$. Consider the following convex functional

$$f(y_1,...,y_n) = \log(q(e^{y_1}, e^{y_2}, ..., e^{y_n})).$$

Here $q(x), x \in R^n$ is a homogeneous polynomial of degree n in n real variables with nonnegative coefficients . Then

$$\frac{\alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1,...,\alpha_n)}{q(\alpha_1,...,\alpha_n)} = \frac{\partial}{\partial y_i} f(y_1,...,y_n), 1 \leq i \leq n.$$

Notice the condition (3) is equivalent to the following condition :

$$\inf_{y_1+...+y_n=0} f(y_1,...,y_n) = L > -\infty.$$

Consider the anti-gradient flow , i.e. the system of differential equations

$$y_i(t)' = -(\frac{\partial}{\partial y_i} f(y_1,...,y_n) - 1), y_i(0) = 0; 1 \leq i \leq n.$$

It is well known that in this convex case the gradient flow is defined for all $t \geq 0$. Using the Euler's identity we get that

$$\frac{d}{dt} f(y_1(t),...,y_n(t)) = -\beta(t) =: - \sum_{1\leq i\leq n} |\frac{\alpha_i \frac{\partial}{\partial \alpha_i} q(\alpha_1,...,\alpha_n)}{q(\alpha_1,...,\alpha_n)} - 1|^2$$

It is easy to see that , because of the convexity of f , a nonnegative function $\beta(t)$ is non-increasing on $[0,\infty)$.
As $\inf_{y_1+...+y_n=0} f(y_1,...,y_n) = L > -\infty$ thus $\int_0^\infty \beta(t)dt < \infty$. Thus $\lim_{t\to\infty}\beta(t) = 0$. This proves the implication (3) $\rightarrow$ (4) for all homogeneous polynomials of degree n in n real variables with nonnegative coefficients .
The implication (4) $\rightarrow$ (5) is obvious . The implication (5) $\rightarrow$ (6) for general homogeneous polynomials of degree n in n real variables with nonnegative coefficients is Proposition 2.6 .
Finally , the implication (6) $\rightarrow$ (2) follows fairly directly from Theorem 2.2 . ∎

D Lower bounds on the number of oracle calls for the exact computation of $\frac{\partial^n}{\partial x_1 ... \partial x_n} p(x_1,...,x_n)$

Definition D.1: Call a set $\{X_1,...,X_m\}, X_i \in C^n$ ϵ-universal if there exist complex numbers $c_1,...,c_m$ such that for any homogeneous polynomial $p(.)$ of degree n in n complex variables the following inequality holds

$$\begin{aligned}&|\tfrac{\partial^n}{\partial x_1 ... \partial x_n} p(x_1,...,x_n) - \textstyle\sum_{1\leq i\leq m} c_i p(X_i)| \\ &\quad \leq \epsilon \max_{(r_1,...,r_n)\in I_{n,n}} |a_{r_1,...,r_k}|,\end{aligned} \tag{28}$$

where $a_{r_1,...,r_n}, (r_1,...,r_n) \in I_{n,n}$ are the coefficients of the polynomial $p(.)$. ∎

Lemma D.2: *If the set* $\{X_1,...,X_m\}, X_i \in C^n$ *is* 0*-universal then*

$$m \geq \frac{n!}{[\frac{n}{2}]!(n-[\frac{n}{2}])!} \approx \frac{2^n}{\sqrt{n}} \tag{29}$$

If the set $\{X_1,...,X_m\}, X_i \in C^n$ *is* ϵ*-universal then*

$$m \geq \min([\frac{1}{\epsilon}], \frac{n!}{[\frac{n}{2}]!(n-[\frac{n}{2}])!}) \tag{30}$$

Proof: Define a monomial $M_{r_1,...,r_n}(x_1,...,x_n) = x_1^{r_1} x_2^{r_2} ... x_n^{r_n}$. As $\{X_1,...,X_m\}$ is universal thus the exists complex numbers $(c_1,...,c_m)$, which are wlog are all nonzero , such that

$$\sum_{1\leq i\leq m} c_i M_{r_1,...,r_n}(c_i^{\frac{1}{n}} X_i) = 0$$

if $(r_1,...,r_n) \in I(n,n), (r_1,...,r_n) \neq (1,1,...,1)$;
and $\sum_{1\leq i\leq m} M_{1,1,...,1}(c_i^{\frac{1}{n}} X_i) = 1$; define $Y_i = c_i^{\frac{1}{n}} X_i$ (here $c_i^{\frac{1}{n}}$ is one of the nth complex roots of c_i).
Let $Half = \{(r_1,...,r_n) : r_i \in \{0,1\}, \sum_{1\leq i\leq n} r_i = [\frac{n}{2}]$. Notice that the cardinality $|Half| = \frac{n!}{[\frac{n}{2}]!(n-[\frac{n}{2}])!} =: K$.
Define the following two $K \times m$ complex matrices :

$$W((r_1,...,r_n), j) = M_{r_1,...,r_n}(Y_j),$$

$$V((r_1,...,r_n),j) = M_{1-r_1,...,1-r_n}(Y_j) :$$
$$(r_1,...,r_n) \in Half, 1 \leq j \leq m.$$

Clearly , $Rank(W) = Rank(V) \leq m$. On the other hand the 0-universality condition implies the matrix identity
$$WV^T = I \tag{31}$$
Therefore $m \geq Rank(W) \geq |Half| = \frac{n!}{[\frac{n}{2}]!(n-[\frac{n}{2}])!}$.
If the set $\{X_1,...,X_m\}$ is ϵ-universal and $d\epsilon < 1, d \in N$ then $Rank(WV^T) \geq d$. This proves (30) . ∎

Remark D.3: The identity (7) is a particular case of a slightly more general one :
$$\frac{\partial^n}{\partial x_1 ... \partial x_N} p(x_1,...,x_n) = E(p(z_1,z_2,...,z_n) \prod_{1 \leq i \leq n} \overline{z_i}), \tag{32}$$
where $(z_1,z_2,...,z_n)$ are independent complex random variables such that $E(z_i) = 0$ and $E(z_i\overline{z_i}) = 1$ for all $1 \leq i \leq n$. The identity is easily proved by checking it for all monomials $M_{r_1,...,r_n}, (r_1,...,r_n) \in I(n,n)$. If $p(.)$ is a multilinear polynomial , i.e.
$$p(x_1,...,x_n) = \prod_{1 \leq i \leq n} (\sum_{1 \leq j \leq n} A(i,j)x_j),$$
then $\frac{\partial^n}{\partial x_1 ... \partial x_N} p(x_1,...,x_n) = Per(A)$, where $per(A)$ is the permanent of the matrix A . Clearly , lower bound $m \geq \frac{n!}{[\frac{n}{2}]!(n-[\frac{n}{2}])!}$ also holds for multilinear polynomials and even for powers $(\sum_{1 \leq i \leq n} a_i x_i)^n$. It is very likely that the actual lower bound is 2^{n-1} and that it does exist somewhere in the geometrical designs literature . In the case of permanents , the formula (7) is essentially the Ryser's formula [4]; and Lemma D.2 says that , in some sense , it is an optimal formula for computing permanents .
Another equivalent formulation of Lemma D.2 is the following statement :
Let a set of complex vectors
$$S = \{X_l = (x_{l,1},...,x_{l,n}) \in C^n : 1 \leq l \leq \frac{(2n-1)!}{(n-1)!n!}\}$$
be a Haar set for the monomials $M_{r_1,...,r_n} : (r_1,...,r_n) \in I_{n,n}$.
I.e. the square matrix $\{M_{r_1,...,r_n}(X_i) : X_i \in S; (r_1,...,r_n) \in I_{n,n}\}$ is nonsigular .
If
$$\prod_{1 \leq i \leq n} x_{l,i} = \sum_{1 \leq k \leq m} c_i (\sum_{1 \leq i \leq n} Y(k,i)x_i)^n$$
for all $1 \leq l \leq \frac{(2n-1)!}{(n-1)!n!}$ and some complex numbers $\{c_k; Y(k,i) : 1 \leq k \leq m, 1 \leq i \leq n\}$ then
$$m \geq \frac{n!}{[\frac{n}{2}]!(n-[\frac{n}{2}])!} \approx \frac{2^n}{\sqrt{n}}.$$

∎